Lecture 36

$\boxed{\text{E.g.}}$

$$A = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix}$$

Diagonalize A

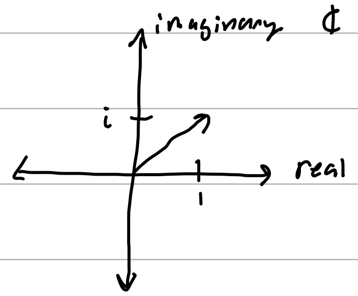
E-vals: $0 = \det(A - \lambda I)$

$$0 = \begin{vmatrix} 3-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) - (-5)$$

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \left(\sqrt{(-2)^2 - 4(1)(2)}\right)}{2} = 1 \pm \sqrt{-1} = 1 \pm i$$

$1 + i \in \mathbb{C}$ (complex numbers)



e-vecs:

$\lambda = 1+i$ so $\left[A - (1+i)I \mid \vec{0} \right] = \left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -2-i & 0 \\ 0 & 0 & 0 \end{array}\right]$

$$\text{e-}\vec{\text{vec}} = \vec{v} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix} \in \mathbb{R}^2$$

for $\lambda = 1+i$

For $\lambda = 1-i$, use "complex conjugation"

$\lambda = a+bi \in \mathbb{C}$ $\quad \bar{\lambda} := a-bi$

$\boxed{ex}$ $\lambda = 1+i$ $\quad \bar{\lambda} = 1-i$

Also: if $A \in M_{n \times n}$ real matrix and $A\vec{v} = \lambda \vec{v}$ $\rightarrow$ $\overline{Av} = \overline{\lambda v}$

$\rightarrow \bar{A}\bar{v} = \bar{\lambda}\bar{v}$

Write: $v = v_1 + i v_2$ $\quad \rightarrow A\bar{v} = \bar{\lambda}\bar{v}$

$\bar{v} = v_1 - i v_2$

so $\bar{v}$ is an e-vec with e-val $\bar{\lambda}$.

Back to eg

e-vals	$1+i$	$1-i$
evec	$\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$	$\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

Basis β for $\mathbb{C}^2$

$$\begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+i & 2-i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \begin{bmatrix} 2+i & 2-i \\ 1 & 1 \end{bmatrix}^{-1}$$

Alternative: instead of using basis $\beta = \{\vec{v}_1, \vec{v}_2\}$

- use basis $\beta_{\mathbb{R}} = \{\vec{v}_2, \vec{v}_1\}$ where $\vec{v}_1$ and $\vec{v}_2$ are real and imaginary parts of $\vec{v}$.

- In e.g. $\beta_{\mathbb{R}} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

Q: If $T(\vec{v}) = A\vec{v}$ then what is $[T]_{\beta_{\mathbb{R}}}$?

ANS: $[T]_{\beta_{\mathbb{R}}} = \begin{bmatrix} [T(v_2)]_{\beta_{\mathbb{R}}} & [T(v_1)]_{\beta_{\mathbb{R}}} \end{bmatrix}$

$$Av = \lambda v$$

$$\rightarrow A(v_1 + iv_2) = (a+bi)(v_1 + iv_2)$$

$$\rightarrow Av_1 + iAv_2 = (av_1 - bv_2) + i(bv_1 + av_2)$$

$$\rightarrow \begin{cases} Av_1 = av_1 - bv_2 \\ Av_2 = bv_1 + av_2 \end{cases}$$

$$[T]_{\beta_{\mathbb{R}}} = \begin{bmatrix} [bv_1 + av_2]_{\beta_{\mathbb{R}}} & [av_1 - bv_2]_{\beta_{\mathbb{R}}} \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

so $A = \begin{bmatrix} v_2 & v_1 \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} v_2 & v_1 \end{bmatrix}^{-1}$

"$S \, [T]_{\beta_{\mathbb{R}}} \, S^{-1}$"

$\boxed{\text{In Eg}}$

$$\begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$$

This is rotation and scaling matrix

$$\begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix} = (\sqrt{2}) \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{\text{rotation matrix}} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$$

Scaler part

rotation matrix